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LETTER TO THE EDITOR

Fractional integral and differential equations for a class of Lévy-type probability densities

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Abstract. An application of the fractional calculus to a class of Lévy distribution functions leads to the conclusion that the Lévy index (fractal dimension) μ is identical to the order of the fractional Liouville–Riemann integral operator. The corresponding fractional integral and differential equations will be presented and solutions of Lévy-type, one-sided probability densities will be given and discussed.

Stable distributions play a dominant part in the theory of random variables. Non-standard random walks, for instance, which do not have a fixed step size, but which are based on a variable step size x, are usually described by a one-sided probability distribution function of asymptotic type (large x-values)

$$f(x) \sim x^{-1-\mu} \qquad \mu > 0 \qquad x > 0 \tag{1}$$

and are called Lévy flights since (1) represents a Lévy distribution [1], and the trace of the sites visited by the walker forms a set of fractal dimension μ [2]. Here, we will show that for a certain class of one-sided probability densities $f(x)$, defined on R_+, the Lévy-index μ can be related to the order of a fractional integral operator. For this reason we apply the fractional calculus [3] based on the Liouville–Riemann definition for the *fractional integral operator* ${}_0D_x^{-q}$ given by

$${}_0D_x^{-q}f(x) := \frac{1}{\Gamma(q)} \int_0^x (x-y)^{q-1} f(y)\,\mathrm{d}y \qquad q > 0. \tag{2}$$

In general, q is a non-integer number. If $q = 1, 2, 3 \ldots$ then (2) is just the standard Riemann integral. The *fractional differential operator* ${}_0D_x^{\nu}$ for $\nu > 0$ is given by the definition

$${}_0D_x^{\nu}f(x) := \frac{\mathrm{d}^n}{\mathrm{d}x^n}\left({}_0D_x^{\nu-n}f(x)\right) \qquad \nu - n < 0 \tag{3}$$

where ${}_0D_x^{\nu-n}$ for $\nu - n < 0$ is defined in (2), indicating that for diffintegrable (i.e. differentiable and integrable) functions $f(x)$ the operation 'fractional differentiation' can be decomposed into a fractional integration followed by an ordinary differentiation $\mathrm{d}^n/\mathrm{d}x^n$. Here, n is the least positive integer greater than ν. If $0 < \nu < 1$, then we choose $n = 1$, and if $1 < \nu < 2$ we take $n = 2$ and so on.

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Almost a decade ago it had already been suggested [4] that Lévy-type probability functions are not just solutions of standard-type differential equations but rather are to be represented by integral equations with an integral kernel $K(x-y) \sim (x-y)^{-\alpha}$ of some fractional order α. More recently a Fox-function representation [5] has been given for a class of probability densities $f(x)$ defined by the integral equation

$$x^m f(x) = \int_0^x (x-y)^{-\alpha} f(y)\,\mathrm{d}y \qquad x>0 \tag{4}$$

where $\alpha > 0$ is a non-integer number, and m is a positive integer ($m = 1, 2, 3, \ldots$). The motivation of our investigation is to show that the special class of normalised one-sided Lévy-type probability densities

$$f(x) = \frac{a^\mu}{\Gamma(\mu)} x^{-\mu-1} \exp(-a/x) \qquad a>0 \qquad x>0 \tag{5}$$

is a solution of the fractional integral equation

$$x^{2q} f(x) = a^q {}_0D_x^{-q} f(x) \tag{6}$$

if we identify the Lévy-index μ as the fractional order q of the integral operator ${}_0D_x^{-q}$. We note that $f(x)$, given in (5), tends to zero for $x \to 0$ and has for large x-values the desired asymptotic behaviour (1). Figure 1 shows $f(x)$ for several values of μ. It is obvious that the asymptotic power-law tail (1) satisfies the scaling property $f(\lambda x) = \lambda^{-\mu-1} f(x)$ where λ is the scaling factor, and μ has been identified as a fractal (similarity) dimension [2]. Self-similar scaling processes based on Lévy dynamics play a dominant part in the study of random walks, of ion-channel gating kinetics [6] and in the understanding of irregular structures and pattern formation in biophysical systems [7, 8].

For $a = \pi$ and $q = \frac{1}{2}$, equation (6) leads to the Abel-type integral equation

$$xf(x) = \int_0^x (x-y)^{-1/2} f(y)\,\mathrm{d}y \tag{7}$$

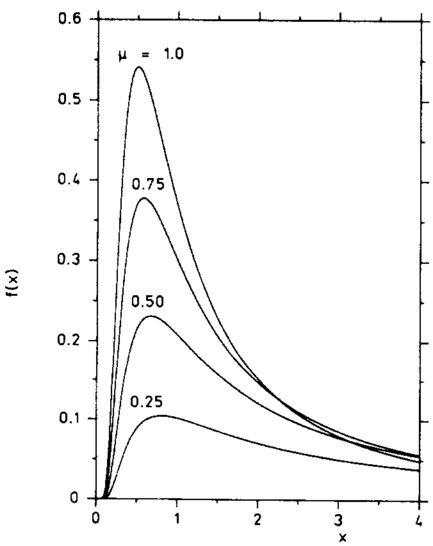

Figure 1. Probability density $f(x)$ (equation (5)) for some values of the Lévy index μ and for $a = 1$.

which has already been solved by Laplace transform techniques [9]. The solution is $f(x)=x^{-3/2}\exp(-\pi/x)$. To our knowledge a solution of (6) for arbitrary non-integer positive q-values has not been given up to now, and a relationship between q and μ has not been pointed out.

In order to show that (5) is a solution of (6) we insert $f(x)$, given in (5), into the integral (6) and substitute $y=ax/(xz+a)$ leading to

$$a^q{}_0D_x^{-q}f(x)=\frac{a^q}{\Gamma(q)}\frac{a^\mu}{\Gamma(\mu)}\int_0^x(x-y)^{q-1}y^{-\mu-1}\,\mathrm{e}^{-a/y}\,\mathrm{d}y$$

$$=\frac{a^q}{\Gamma(q)}\frac{\mathrm{e}^{-a/x}}{x^{\mu+1-2q}}\frac{1}{\Gamma(\mu)}\int_0^x\frac{z^{q-1}\,\mathrm{e}^{-z}\,\mathrm{d}z}{(xz+a)^{\mu-q}}.$$

For $q=\mu$ the remaining integral is just Euler's definition of the Γ-function $\Gamma(\mu)$ for $\mu>0$. Thus, we have found for $q=\mu$:

$$a^\mu{}_0D_x^{-\mu}f(x)=\frac{a^\mu}{\Gamma(\mu)}\frac{\mathrm{e}^{-a/x}}{x^{1-\mu}}=x^{2\mu}f(x)$$

which is identical with the left-hand side of (6) if we take there $q=\mu$, which completes our proof.

We note that the class of probability densities (5) is non-negative, and has the moments ($k=0,1,2,\ldots$)

$$\langle x^k\rangle=\int_0^\infty x^kf(x)\,\mathrm{d}x=a^k\Gamma(\mu-k)/\Gamma(\mu) \qquad (8)$$

including normalisation ($\langle x^0\rangle=1$).

Making use of the definition (3) for fractional differentiation we find in a similar way, for $n-\nu=\mu>0$, the following *fractional differential equation* for the class $f(x)$ of Lévy-type probability densities (5):

$${}_0D_x^\nu f(x)={}_0D_x^{n-\mu}f(x)=a^{-\mu}\frac{\mathrm{d}^n}{\mathrm{d}x^n}(x^{2\mu}f(x)). \qquad (9)$$

Here, one has to take $n=1$ (i.e. $\nu=1-\mu$) if $0<\mu<1$ and $n=2$ (i.e. $\nu=2-\mu$) if $1<\mu<2$ etc. In physical applications of Lévy distributions one is mainly to be concerned with the cases $0<\mu<1$ and $1<\mu<2$.

The fractional calculus is old but little studied. However, in the last decade, some investigators have discussed a few interesting problems concerning diffusion processes in media with fractal geometry [3, 10], and have formulated fractional diffusion [11] and fractional Boltzmann equations [12].

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